On the behavior of bounded vorticity, bounded velocity solutions to the 2D Euler equations

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Bounded solutions to the Euler equations

Let $\boldsymbol{u}_0$ be a divergence-free vector field in $L^\infty(\mathbb{R}^2)$ having bounded vorticity. It is shown (more-or-less) by Serfati in 1995 (made precise by Ambrose, K, Lopes Filho, Nussenzveig Lopes 2013) that there exists a unique solution, $\boldsymbol{u}$, to the Euler equations such that:

- $\boldsymbol{u} \in C([0, T]; L^\infty)$ is divergence-free, $\omega := \operatorname{curl} \boldsymbol{u} \in L^\infty([0, T] \times \mathbb{R}^2)$;
- $\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = 0$ as distributions;
- vorticity, ω, is transported by the classical flow map.
- For any radially symmetric, smooth, compactly supported cutoff function a with $a = 1$ in a neighborhood of the origin, we have

$$
\begin{aligned}
u^j(t) - (u^0)^j &= (aK^j) * (\omega(t) - \omega^0) \\
&\quad - \int_0^t \left(\nabla \nabla^\perp \left[(1 - a) K^j \right] \right) *\cdot (u \otimes u)(s) \, ds.
\end{aligned}
$$

Goal 1: Remove the need for this strange identity.

2D Euler: a disturbing example

Let $\boldsymbol{U}_\infty$ be an arbitrary, smooth vector-valued function of time and set

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{U}_\infty(t), \quad p(t,\boldsymbol{x}) = -\boldsymbol{U}'_\infty(t)\cdot\boldsymbol{x}.$$

Then $\operatorname{div}\boldsymbol{u} = 0$ and

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{U}'_\infty(t) - \boldsymbol{U}'_\infty(t) = 0,$$

so $(\boldsymbol{u}, p)$ satisfy 2D Euler (and Navier-Stokes).

But,

$$\boldsymbol{u}(t,x) = \boldsymbol{U}_\infty(0), \quad p(t,x) = 0$$

solve the same equations with the same initial velocity, $\boldsymbol{U}_\infty(0)$.

Jun Kato [2003]: this example shows that that the behavior at infinity of the pressure must be constrained (to sublinear growth in his case) to insure uniqueness of [Euler or] Navier-Stokes.

The two examples are actually the same

Change to an accelerated frame of reference by the transformation,

$$\overline{x} = \overline{x}(t, x) = x + \int_0^t \boldsymbol{U}_\infty(s)\, ds,$$

$$\overline{u}(t, x) = u(t, \overline{x}) - \boldsymbol{U}_\infty(t), \quad \overline{p}(t, x) = p(t, \overline{x}) + \boldsymbol{U}'_\infty(t) \cdot \overline{x}.$$

(This is a Galilean transformation when $\boldsymbol{U}_\infty$ is constant in time.)

One can show that this transforms the first of the two solutions into the second solution—in the transformed frame. In some sense, then, the first solution is expressed in an accelerated reference frame, while the second is expressed in an inertial frame.

Satisfying Goal 1

To satisfy Goal 1, we will show that any solution to the Euler equations satisfying our first three, natural, criteria must satisfy the fourth (ugly identity) after possibly being translated to an inertial frame.

The low regularity of our solutions is mostly just a troublesome technical issue; the key issue is the lack of decay of the solutions.

Goal 2

Serfati 1995 claims that his uniqueness criteria is sublinear growth of the pressure, but that is not what he (more-or-less) proves. Taniuchi, Tashiro, and Yoneda 2010 do use sublinear growth of the pressure as their criteria, but it is not immediately clear, then, that their solutions are the same as Serfati's.

Goal 2: Obtain properties of the pressure for bounded solutions and show that sublinear growth yields the solutions of Serfati's: that is, show that sublinear growth of the pressure selects for the inertial frame. In addition, show that Serfati's and Taniuchi's solutions are the same.

Goal 3

Goal 3: Accomplish as much as possible of Goals 1 and 2 for an exterior domain (specifically, the exterior to a single obstacle).

We will find no simple change of variables that will eliminate the pressure for an exterior domain: there is no preferred reference frame.

I will focus on discussing Goals 1 and 2.

Decaying vorticity

Consider, first, the case of compactly supported vorticity, ω. We know that the Biot-Savart law,

$$\boldsymbol{v} = \boldsymbol{K} * \omega, \qquad \boldsymbol{K}(\boldsymbol{x}) = \frac{1}{2\pi} \frac{\boldsymbol{x}^\perp}{|\boldsymbol{x}|^2} = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|\boldsymbol{x}|^2},$$

gives a divergence-free vector field, $\boldsymbol{v}$, whose vorticity is ω. Moreover, $\boldsymbol{v}$ decays to zero at infinity.

If $\boldsymbol{u}$ is the actual solution having vorticity ω then $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same divergence and same vorticity. As long as they are tempered distributions, it follows that they differ by an harmonic polynomial. Since velocity is to be bounded and continuous, it follows that

$$\boldsymbol{u} = \boldsymbol{U}_\infty(t) + \boldsymbol{K} * \omega$$

for some $\boldsymbol{U}_\infty \in C([0, T])^2$. (And so the velocity at infinity can be made well-defined for compactly supported vorticity.)

Reliance on the Biot-Savart law

This argument never used the Euler equations themselves, but relied totally upon the ready availability of a method for recovering a velocity from the vorticity. For non-decaying vorticity this ready method is lost.

We need a replacement.

Define the *Serfati space*,

$$S = S(\mathbb{R}^2) = \left\{ \boldsymbol{u} \in (L^\infty(\Omega))^2 \colon \operatorname{div} \boldsymbol{u} = 0, \omega(\boldsymbol{u}) \in L^\infty \right\},$$

$$\|u\|_S = \|\boldsymbol{u}\|_{L^\infty} + \|\omega(\boldsymbol{u})\|_{L^\infty}.$$

For $\boldsymbol{u} \in S$, $\boldsymbol{K} * \omega$ does not exist as an absolutely convergent integral. If we cutoff $\boldsymbol{K}$, however, we would obtain an absolutely convergent integral. This leads us to examine the sequence, $\left((a_R \boldsymbol{K}) * \omega(\boldsymbol{u}) \right)_{R=1}^\infty$, where a_R is a cutoff function with increasing support.

Renormalized Biot-Savart law

Let a be a radially symmetric, smooth, compactly supported function with $a = 1$ in a neighborhood of the origin. We will refer to such a function simply as a **radial cutoff function.** For any $R > 0$ we define

$$a_R(\cdot) = a(\cdot/R).$$

We will find that:

Proposition (Renormalized Biot-Savart law)

For any $\boldsymbol{u} \in S$ there exists a constant vector field, $\boldsymbol{H}$, and a subsequence, (R_k), $R_k \to \infty$, such that

$$\boldsymbol{u} = \boldsymbol{H} + \lim_{k\to\infty} (a_{R_k}\boldsymbol{K}) * \omega(\boldsymbol{u}),$$

convergence being uniform on compact subsets.

Radial symmetry of a simplifies proofs, but is not essential.

Characterization of 2D Euler in the full plane

Theorem

Let $\boldsymbol{u}$ be a solution to Euler in the full plane. There exists a continuous vector-valued function of time, $\boldsymbol{U}_\infty$, with $\boldsymbol{U}_\infty(0) = 0$, for which

$$\boldsymbol{u}(t, \boldsymbol{x}) - \boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{U}_\infty(t) + \lim_{R \to \infty} (a_R \boldsymbol{K}) * (\omega(t) - \omega(0))(\boldsymbol{x}).$$

There exists a pressure, p, for which $\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0$ as distributions, with

$$\begin{aligned} \nabla p(t, \boldsymbol{x}) &= -\boldsymbol{U}'_\infty(t) + O(1), \\ p(t, \boldsymbol{x}) &= -\boldsymbol{U}'_\infty(t) \cdot \boldsymbol{x} + O(\log |\boldsymbol{x}|). \end{aligned}$$

Moreover, given any such $\boldsymbol{U}_\infty$ $\exists!$ solution, $(\boldsymbol{u}, p)$, as above.

- $\boldsymbol{U}'_\infty$ is a distributional derivative on $(0, T)$.

The Serfati identity

The velocity expression,

$$\boldsymbol{u}(t) - \boldsymbol{u}(0) = \boldsymbol{U}_\infty(t) + \lim_{R \to \infty} (a_R \boldsymbol{K}) * (\omega(t) - \omega(0)),$$

is the renormalized Biot-Savart law applied to $\omega(t) - \omega(0)$ *without the need to take a subsequence*.

We will find that this is equivalent to the Serfati identity,

$$\begin{aligned} \boldsymbol{u}(t) - \boldsymbol{u}(0) &= \boldsymbol{U}_\infty(t) + (a\boldsymbol{K}) * (\omega(t) - \omega(0)) \\ &\quad - \int_0^t \left(\nabla \nabla^\perp [(1-a)\boldsymbol{K}] \right) *\cdot (\boldsymbol{u} \otimes \boldsymbol{u})(s)\, ds. \end{aligned}$$

Ihis identity follows formally by applying a cutoff function to the vorticity equation and integrating by parts. We cannot do this for a bounded solution because of a lack of decay, but one can show that if the Serfati identity holds for one cutoff function it holds for any other. The issue is regularity (which we have just enough of) not lack of decay.

Outline of proof of Characterization

Highest level:

1. Characterize the velocity; that is, prove that for any solution, the velocity expression must hold.
2. Prove existence and uniqueness of solutions, given any $\boldsymbol{U}_\infty$, by employing a sequence of smooth solutions having compact vorticity. Requires only a slight modification of the proof in Ambrose, K, Nussenzveig Lopes, Lopes Filho, which we refer to as AKLL.
3. Prove that the solutions constructed in step 2 satisfy the pressure relations by taking a limit for the approximating sequence.

Characterization of velocity

Let $\boldsymbol{u}$ be a solution to 2D Euler in the full plane.

1. Show that the Serfati identity,

$$\begin{aligned}\boldsymbol{u}(t) - \boldsymbol{u}(0) &= \boldsymbol{U}_\infty(t) + (a\boldsymbol{K}) * (\omega(t) - \omega(0)) \\ &\quad - \int_0^t \left(\nabla\nabla^\perp [(1-a)\boldsymbol{K}]\right) *\cdot (\boldsymbol{u} \otimes \boldsymbol{u})(s)\, ds,\end{aligned}$$

   implies the velocity expression,
   $\boldsymbol{u}(t) - \boldsymbol{u}(0) = \boldsymbol{U}_\infty(t) + \lim_{R\to\infty} (a_R\boldsymbol{K}) * (\omega(t) - \omega(0))$.

2. Conversely, show that if the velocity expression holds for a *subsequence* (which is allowed to vary with time) then the Serfati identity holds.
3. Prove that the renormalized Biot-Savart law holds $\forall\ \boldsymbol{v} \in S$.
4. Let $\boldsymbol{v} = \boldsymbol{u}(t) - \boldsymbol{u}^0$. Then the velocity expression holds for a subsequence by step 3, so the Serfati identity holds by step 2, so the velocity expression holds for the full sequence by Step 1.

Renormalized Biot-Savart law

Proposition (Renormalized Biot-Savart law)

For any $\boldsymbol{u} \in S$ *there exists a constant vector field,* $\boldsymbol{H}$*, and a subsequence,* (R_k)*,* $R_k \to \infty$*, such that*

$$\boldsymbol{u} = \boldsymbol{H} + \lim_{k \to \infty} (a_{R_k} \boldsymbol{K}) * \omega(\boldsymbol{u}),$$

convergence being uniform on compact subsets.

The proof involves finding just enough compactness to obtain a convergent subsequence.

The pressure

The velocities of the approximate sequence of solutions decay sufficiently rapidly at infinity, so we can relatively easily deal with their associated pressures. Then, we take a limit and show that the bounds we obtain hold in the limit.

There are two key ideas in bounding the pressures for the approximate solutions. First is an identity from a 1995 paper of Serfati:

$$\nabla p(\boldsymbol{x}) = -\boldsymbol{U}'_\infty + \int_{\mathbb{R}^2} a(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{K}^\perp(\boldsymbol{x}-\boldsymbol{y}) \;\operatorname{div}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{y}) \;d\boldsymbol{y}$$
$$+ \int_{\mathbb{R}^2} (\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{y}}\left[(1-a(\boldsymbol{x}-\boldsymbol{y}))\boldsymbol{K}^\perp(\boldsymbol{x}-\boldsymbol{y})\right] \;d\boldsymbol{y}.$$

But, $\operatorname{div}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = \nabla\boldsymbol{u}\cdot(\nabla\boldsymbol{u})^T \in L^p$ for all $p \in [1,\infty)$ while $a\boldsymbol{K} \in L^q$ for all $q \in [1,2)$, and $\nabla\nabla\left[(1-a)\boldsymbol{K}\right] \in L^1$ while $\boldsymbol{u}\otimes\boldsymbol{u} \in L^\infty$ can be used to show that $\nabla p + U'_\infty \in L^\infty([0,T]\times\Omega)$.

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$$+ \int_{\mathbb{R}^2} (\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{y}} \left[(1-a(\boldsymbol{x}-\boldsymbol{y}))\boldsymbol{K}^\perp(\boldsymbol{x}-\boldsymbol{y})\right] \ d\boldsymbol{y}.$$

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Dini MOC

The second key idea never introduces a cutoff function. Instead, one writes the pressure as a Riesz transform of $\boldsymbol{u} \otimes \boldsymbol{u}$. To describe it, we need a definition.

Definition

A nondecreasing continuous function, $\mu\colon [0, \infty) \to [0, \infty)$, is a modulus of continuity (MOC) if $\mu(0) = 0$ and $\mu > 0$ on $(0, \infty)$. Given a MOC, μ, we define,

$$S_\mu(x) = \int_0^x \frac{\mu(r)}{r} \, dr.$$

We say that μ is *Dini* if S_μ is finite for some (and hence all) $x > 0$.

Using a Riesz transform

The following lemma is as in Kiselev, Nazarov, Volberg [2007]. (This type of bound goes back to Burch [1978] for a singular integral operator in a bounded domain.)

Lemma

Let R be any Riesz transform. Suppose that h lying in $L^p(\mathbb{R}^2)$ for some p in $[1, \infty)$ has a concave Dini MOC, μ. Then Rh has a MOC,

$$\nu(r) = C \left(S_\mu(r) + r \int_r^\infty \frac{\mu(s)}{s^2} \, ds \right)$$

for some absolute constant, C. (Note that this MOC holds for all $r > 0$.)

Then $p = R(\boldsymbol{u} \otimes \boldsymbol{u})$, and $\boldsymbol{u} \otimes \boldsymbol{u}$ Has a log-Lipschitz MOC. The resulting MOC, ν, can be used to control p asymptotically in $|\boldsymbol{x}|$ by fixing the value of p at one point. The resulting bound is $C \log |\boldsymbol{x}|$ for large $\boldsymbol{x}$.

The exterior to a single obstacle

Letting Ω be the exterior to a single, connected and simply connected obstacle with $C^{2,\alpha}$ boundary, we obtain analogous results.

Theorem

Let $\boldsymbol{u}$ be a solution to Euler in the full plane. There exists a continuous harmonic vector field, $\boldsymbol{U}$, with $\boldsymbol{U} = 0$ at time zero, and a pressure, p, for which

$$\begin{aligned}\boldsymbol{u}(t,x) - \boldsymbol{u}^0(x) &= \boldsymbol{U}(t,x) + \lim_{R\to\infty} \int_\Omega a_R(x-y) J_\Omega(x,y)\omega(y)\,dy,\\ \nabla p(t,x) &= -\partial_t \boldsymbol{U}(t,x) + O(1),\\ p(t,x) &= -\partial_t \zeta(t,x) + O(\log|x|).\end{aligned}$$

Here, J_Ω is the hydrodynamic Biot-Savart kernel and $\boldsymbol{U}$ is defined uniquely by its value, $\boldsymbol{U}_\infty$, at infinity and its circulation, γ, about the boundary. The vector field, ζ, and so the pressure is multi-valued (unless $\gamma = 0$) with $\nabla\zeta = \boldsymbol{U}$.

Difficulties with an exterior domain

The proofs are much more technical because:

- The Biot-Savart kernel is replaced with $K_\Omega = \nabla^\perp G_\Omega$, where G_Ω is the Green's function for the Dirichlet Laplacian on Ω.
- Convolutions are now integrals, so switching the derivatives between terms of a convolution becomes integration by parts. Because $a_R K$ is no longer a compactly supported distribution, these need to be justified.
- The estimates on K_Ω are no longer immediate and are quite involved. Fortunately, the estimates were done in AKLL.
- We must strengthen the definition of a solution somewhat to allow integration by parts in the proof that the Serfati identity is independent of the cutoff function.

Handling pressure in an exterior domain

- The equation for the pressure involves Neumann boundary conditions. We define a Neumann function N_Ω (Green's function of the second kind) for Ω and obtain interior elliptic regularity estimates using it to replace the Riesz transform approach in the full plane.
- Fortunately, the estimates on K_Ω in AKLL can be used to obtain many of the necessary estimates on N_Ω.
- A recent paper by Nardi on elliptic regularity with Neumann boundary conditions in a bounded domain is also helpful.

Related work: Existence

I have mentioned the work of AKLL, upon which I have built.

In the full plane, Taniuchi [2004] constructs bounded solutions (actually, "slightly unbounded" vorticity, a localized version of Yudovich [1995]):

- Uses a mild-solution formulation of the Euler equations.
- For his solutions, $\boldsymbol{U}_\infty \equiv 0$.
- He employs an approximate sequence of smooth solutions with non-decaying vorticity coming from another 1995 paper of Serfati.
- Uses Littewood-Paley decompositions but no paradifferential calculus.
- Approach would not extend to an exterior domain because the existence of smooth solutions in an exterior domain having non-decaying vorticity is not known.

Related work: Uniqueness

In the full plane, Taniuchi, Tashiro, and Yoneda [2010] (TTY2010) establish uniqueness of the solutions constructed in Taniuchi [2004]:

- The selection criteria for uniqueness is the sublinear growth of pressure at infinity.
- Employs paradifferential calculus, with an approach to uniqueness (actually, a type of continuous dependence on initial data) adapted from Vishik 1999.
- Does not assume vorticity is transported by the flow map.
- The solutions constructed in AKLL are also mild solutions.
- Because we know that AKLL-solutions have sublinear growth of the pressure, the uniqueness result in TTY2010 implies that the bounded solutions in Taniuchi [2004] and AKLL are the same.
- In particular, vorticity is transported by the flow map for Taniuchi's solutions.

Thank you

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